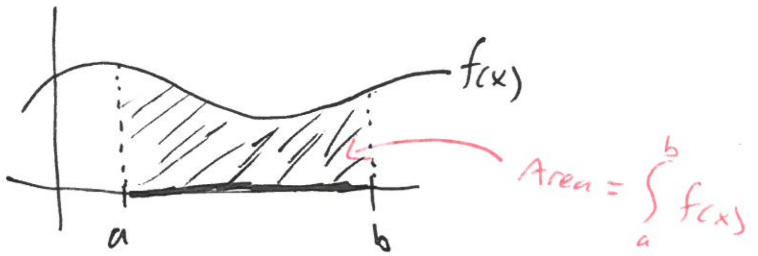
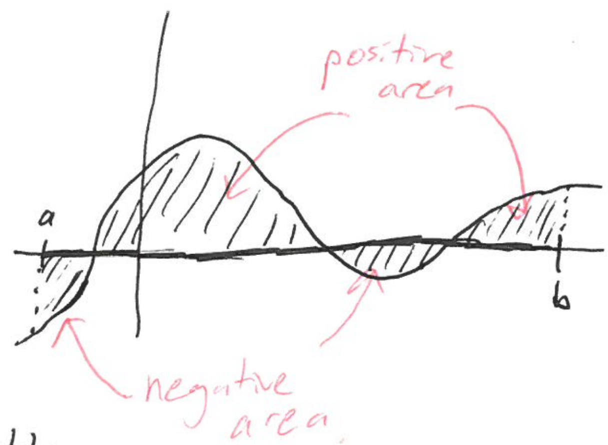
7.6 Double Integrals

Recal how to integrate functions of 1-variable

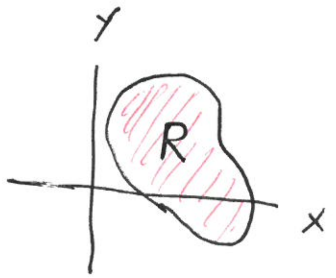
Definite Integral: $\int_a^b f(x)\,dx$

Need:
- The function $f(x)$
- The interval we integrate over $[a, b]$ (from $x=a$ to $x=b$)





For a function of two variables we need:
- The function $f(x,y)$
- The region we integrate over

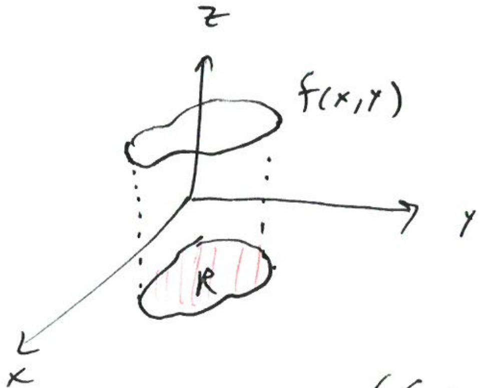


denote $\iint_R f(x,y)\,dx\,dy$

The double integral of $f(x,y)$ over the region R

Suppose $f(x,y) \geq 0$ for all x, y in R

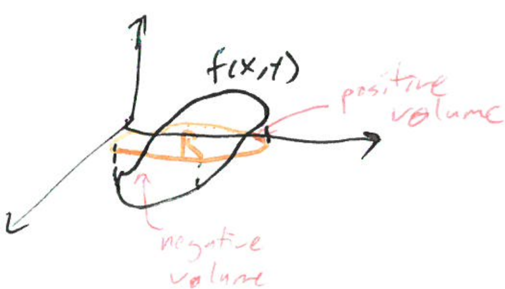
The graph $f(x,y)$ will lie above R in 3-dim space



This will give us a solid bounded above by $f(x,y)$ ~~and~~ over R

$\iint_R f(x,y)\,dx\,dy$ is the volume of this solid

If $f(x,y)$ is negative on R (below the xy plane) we will count that as negative volume



How to calculate the value of a double integral?

- Recall $\int_a^b f(x)\,dx = F(b) - F(a)$

  where $F(x)$ is the antiderivative of $f(x)$

  $(F'(x) = f(x))$

- For $f(x,y)$ consider the iterated integral

$$\int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx$$

and work from inside out

just functions of x

$$\int_{g(x)}^{h(x)} f(x,y)\,d\underline{y}$$

consider $f(x,y)$ as a function of just y

$= F(x, h(x)) - F(x, g(x))$ where $F(x,y)$ is the antiderivative of $f(x,y)$ with respect to y

$(F_y(x,y) = f(x,y))$

we are left with a function in just x and calculate

$$\int_a^b F(x,h(x)) - F(x,g(x))\,dx$$

normally.

ex

$$\int_1^2 \left(\int_3^4 (y-x)\, dy \right) dx$$

starting with inside $\int_3^4 y - x \; dy$

$$= \frac{y^2}{2} - xy \Big]_3^4$$

$$= \left(\frac{16}{2} - 4x\right) - \left(\frac{9}{2} - 3x\right)$$

$$= \frac{16}{2} - 4x - \frac{9}{2} + 3x$$

$$= \frac{7}{2} - x$$

continue with outside

$$\int_1^2 \left(\frac{7}{2} - x\right) dx = \frac{7}{2}x - \frac{x^2}{2} \Big]_1^2$$

$$= \left(\frac{7}{2} \cdot 2 - \frac{4}{2}\right) - \left(\frac{7}{2} - \frac{1}{2}\right)$$

$$= \frac{14}{2} - \frac{4}{2} - \frac{7}{2} + \frac{1}{2}$$

$$= \frac{4}{2} = 2$$

ex

$$\int_0^1 \left(\int_{\sqrt{x}}^{x+1} 2xy\, dy \right) dx$$

Inside:

$$\int_{\sqrt{x}}^{x+1} 2xy\, dy = xy^2 \Big|_{\sqrt{x}}^{x+1}$$

$$= x(x+1)^2 - x\sqrt{x}^2$$

$$= x(x^2+2x+1) - x \cdot x$$

$$= x^3 + 2x^2 + x - x^2$$

$$= x^3 + x^2 + x$$

Outside:

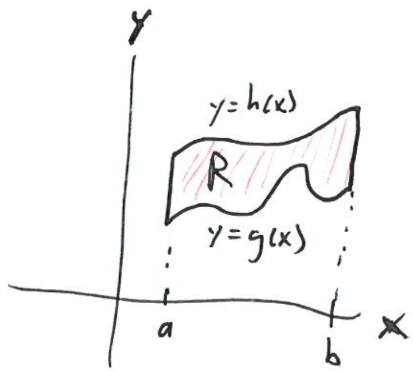
$$\int_0^1 x^3 + x^2 + x\, dx = \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} \Big|_0^1$$

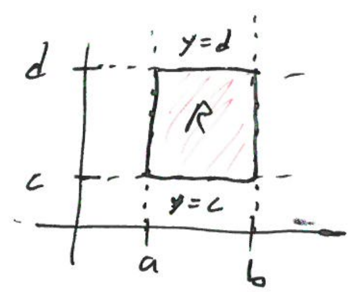
$$= \frac{1}{4} + \frac{1}{3} + \frac{1}{2} = \frac{13}{12}$$

How do we relate $\iint_R f(x,y)\,dx\,dy$ to $\int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx$?

When region R has a special form

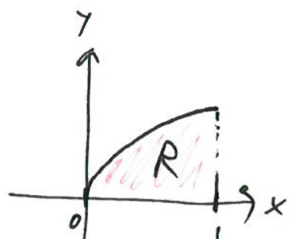
bounded above by a function $h(x)$, below by $g(x)$ and to the left by vertical line $x=a$ and right by $x=b$



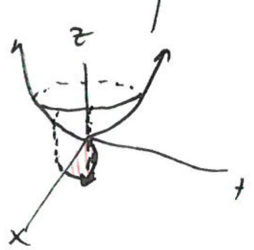


when R is a rectangle like above even easier.

ex/ Calculate the volume of the solid bounded above by $f(x,y) = x^2+y^2$ and below by the xy plane over R, the region bounded by $x=0$, $x=1$, $y=0$, $y=\sqrt[3]{x}$



$$\iint_R f(x,y)\,dx\,dy = \int_0^1 \left(\int_0^{\sqrt[3]{x}} x^2+y^2\,dy \right) dx$$



$$\int_0^{\sqrt[3]{x}} x^2+y^2\,dy = x^2y + \frac{y^3}{3}\Big|_0^{\sqrt[3]{x}}$$

$$= x^2\cdot\sqrt[3]{x} + \frac{\sqrt[3]{x}^{\,3}}{3} - 0$$

$$= x^2\cdot x^{1/3} + \frac{x}{3}$$

$$= x^{7/3} + \frac{x}{3}$$

$$\int_0^1 x^{7/3} + \frac{x}{3}\,dx = \frac{3}{10}x^{10/3} + \frac{x^2}{6}\Big|_0^1$$

$$= \frac{3}{10} + \frac{1}{6}$$

$$= \frac{9}{30} + \frac{5}{30} = \frac{14}{30} = \frac{7}{15}$$

ex

$$\int_{-2}^{0}\left(\int_{-1}^{1} xe^{xy}\,dy\right)dx$$

$$\int_{-1}^{1} xe^{xy}\,dy \qquad \begin{aligned} u &= xy \\ du &= x\,dy \end{aligned} \Rightarrow \int_{y=-1}^{y=1} e^u\,du \Rightarrow e^u\Big|_{y=-1}^{y=1}$$

$$e^{xy}\Big|_{-1}^{1} = e^x - e^{-x} \Rightarrow \int_{-2}^{0} e^x - e^{-x}\,dx$$

$$= e^x + e^{-x}\Big|_{-2}^{0} = e^0 + e^0 - (e^{-2} + e^2) = \boxed{2 - e^{-2} - e^2}$$

ex $$\int_1^4 \left(\int_x^{x^2} x\, y \, dy \right) dx$$

$$\int_x^{x^2} x\, y \, dy = x \frac{y^2}{2} \Big|_x^{x^2} = \frac{x \cdot (x^2)^2}{2} - \frac{x \cdot x^2}{2}$$

$$= \frac{x \cdot x^4}{2} - \frac{x \cdot x^2}{2}$$

$$= \frac{x^5}{2} - \frac{x^3}{2}$$

$$\int_1^4 \frac{x^5}{2} - \frac{x^3}{2} \, dx = \frac{x^6}{12} - \frac{x^4}{8} \Big|_1^4 = \left(\frac{4^6}{12} - \frac{4^4}{8} \right) - \left(\frac{1}{12} - \frac{1}{8} \right)$$

$$= \frac{4^5}{3} - \frac{4^3}{2} + \frac{1}{24} = \frac{2475}{8}$$